X. *On the Motion of Bodies affected by Friction. By the Rev.* Samuel Vince, *A. M. of* Cambridge; *communicated by* Anthony Shepherd, *D. D. F. R. S. Plumian Professor of Astronomy and experimental Philosophy at* Cambridge.

Read November 25, 1784.

THE subject of the paper which I have now the honour of presenting to the Royal Society, seems to be of a very considerable importance both to the practical mechanic and to the speculative philosopher; to the former, as a knowledge of the laws and quantity of the friction of bodies in motion upon each other will enable him at first to render his machines more perfect, and save him in a great measure the trouble of correcting them by trials; and to the latter, as those laws will furnish him with principles for his theory, which when established by experiments will render his conclusions applicable to the real motion of bodies upon each other. But, however important a part of mechanics this subject may constitute, and however, from its obvious uses, it might have been expected to have claimed a very considerable attention both from the mechanic and philosopher, yet it has, of all the other parts of this branch of natural philosophy, been the most neglected. The law by which the motions of bodies are retarded by friction has never, that I know of, been truly established. MUSSCHENBROEK says, that in small velocities the friction varies very nearly as the velocity, but that in great velocities the friction increases; he has also attempted to prove, that by increasing

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the weight of a body the friction does not always increaſe exactly in the ſame ratio; and that the ſame body, if by changing its poſition you change the magnitude of the ſurface on which it moves, will have its quantity of friction alſo changed. HELSHAM and FERGUSON, from the ſame kind of experiments, have endeavoured to prove, that the friction does *not* vary by changing the quantity of ſurface on which the body moves; and the latter of theſe aſſerts, that the friction increaſes very nearly as the velocity; and that by increaſing the weight, the friction is increaſed in the *ſame* ratio. Theſe different concluſions induced me to repeat their experiments, in order to ſee how far they were concluſive in reſpect to the principles deduced from them: when it appeared, that there was another cauſe operating beſides friction, which they had not attended to, and which rendered all their deductions totally inconcluſive. Of thoſe who have written on the theory, no one has eſtabliſhed it altogether on true principles: EULER (whoſe theory is extremely elegant, and which, as he has ſo fully conſidered the ſubject, would have precluded the neceſſity of offering any thing further, had its principles been founded on experiments) ſuppoſes the friction to vary in proportion to the velocity of the body, and its preſſure upon the plane, neither of which are true: and others, who have imagined that friction is a uniformly retarding force (and which conjecture will be confirmed by our experiments), have ſtill retained the other ſuppoſition, and therefore rendered their ſolutions not at all applicable to the caſes for which they were intended. I therefore endeavoured by a ſet of experiments to determine,

1ſt, *Whether friction be a uniformly retarding force.*

2dly, *The quantity of friction.*

3dly,

3dly, *Whether the friction varies in proportion to the pressure or weight.*

4thly, *Whether the friction be the same on whichever of its surfaces a body moves.*

The experiments, in which I was aſſiſted by my ingenious friend the Rev. Mr. JONES, Fellow of Trinity College, were made with the utmoſt care and attention, and the ſeveral reſults agreed ſo very exactly with each other, that I do not ſcruple to pronounce them to be concluſive.

2. A plane was adjuſted parallel to the horizon, at the extremity of which was placed a pulley, which could be elevated or depreſſed in order to render the ſtring which connected the body and the moving force parallel to the plane. A ſcale accurately divided was placed by the ſide of the pulley perpendicular to the horizon, by the ſide of which the moving force deſcended; upon the ſcale was placed a moveable ſtage, which could be adjuſted to the ſpace through which the moving force deſcended in any given time, which time was meaſured by a well regulated pendulum clock vibrating ſeconds. Every thing being thus prepared, the following experiments were made to aſcertain the law of friction. But let me firſt obſerve, that if friction be a uniform force, the difference between it and the given force of the moving power muſt be alſo uniform, and therefore the moving body muſt deſcend with a uniformly accelerated velocity, and conſequently the ſpaces deſcribed from the beginning of the motion muſt be as the ſquares of the times, juſt as when there was no friction, only they will be diminiſhed on account of the friction.

3. EXP. 1. A body was placed upon the horizontal plane, and a moving force applied, which from repeated trials was found to deſcend $52\frac{1}{2}$ inches in $4''$, for by the beat of the clock and

the ſound of the moving force when it arrived at the ſtage, the ſpace could be very accurately adjuſted to the time; the ſtage was then removed to that point to which the moving force would deſcend in 3″, upon ſuppoſition that the ſpaces deſcribed by the moving power were as the ſquares of the times; and the ſpace was found to agree very accurately with the time; the ſtage was then removed to that point to which the moving force ought to deſcend in 2″, upon the ſame ſuppoſition, and the deſcent was found to agree exactly with the time; laſtly, the ſtage was adjuſted to that point to which the moving force ought to deſcend in 1″, upon the ſame ſuppoſition, and the ſpace was obſerved to agree with the time. Now, in order to find whether a difference in the time of deſcent could be obſerved, by removing the ſtage a little above and below the poſitions which correſponded to the above times, the experiment was tried, and the deſcent was always found too ſoon in the former, and too late in the latter caſe; by which I was aſſured that the ſpaces firſt mentioned correſponded exactly to the times. And, for the greater certainty, each deſcent was repeated eight or ten times; and every caution uſed in this experiment was alſo made uſe of in all the following.

EXP. 2. A ſecond body was laid upon the horizontal plane, and a moving force applied which deſcended 41¾ inches in 3″; the ſtage was then adjuſted to the ſpace correſponding to 2″, upon ſuppoſition that the ſpaces deſcended through were as the ſquares of the times, and it was found to agree accurately with the time; the ſtage was then adjuſted to the ſpace correſponding to 1″, upon the ſame ſuppoſition, and it was found to agree with the time.

EXP. 3. A third body was laid upon the horizontal plane, and a moving force applied, which deſcended 59⅝ inches in 4″; the ſtage

ſtage was then adjuſted to the ſpace correſponding to 3″, upon ſuppoſition that the ſpaces deſcended through were as the ſquares of the times, and it was found to agree with the time; the ſtage was then adjuſted to the ſpace correſponding to 2″, upon the ſame ſuppoſition, and it was found to agree with the time; the ſtage was then adjuſted to the ſpace correſponding to 1″, and was found to agree with the time.

Exp. 4. A fourth body was then taken and laid upon the horizontal plane, and a moving force applied, which deſcended 55 inches in 4″; the ſtage was then adjuſted to the ſpace through which it ought to deſcend in 3″, upon ſuppoſition that the ſpaces deſcended through were as the ſquares of the times, and it was found to agree with the time; the ſtage was then adjuſted to the ſpace correſponding to 2″, upon the ſame ſuppoſition, and was found to agree with the time; laſtly, the ſtage was adjuſted to the ſpace correſponding to 1″, and it was found to agree exactly with the time.

Beſides theſe experiments, a great number of others were made with hard bodies, or thoſe whoſe parts ſo firmly cohered as not to be moved *inter ſe* by the friction; and in each experiment bodies of very different degrees of friction were choſen, and the reſults all agreed with thoſe related above; we may therefore conclude, that *the friction of hard bodies in motion is a uniformly retarding force.*

But to determine whether the ſame was true for bodies when covered with cloth, woollen, &c. experiments were made in order to aſcertain it; when it was found in all caſes, that the retarding force increaſed with the velocity; but, upon covering bodies with paper, the conſequences were found to agree with thoſe related above.

4. Having proved that the retarding force of all hard bodies arifing from friction is uniform, the quantity of friction, confidered as equivalent to a weight without inertia drawing the body on the horizontal plane backwards, or acting contrary to the moving force, may be immediately deduced from the foregoing experiments. For let M = the moving force expreffed by its weight; F = *the* friction; W = the weight of the body upon the horizontal plane; S = the fpace through which the moving force defcended in the time t expreffed in feconds; $r = 16\frac{1}{12}$ feet; then the whole accelerative force (the force of gravity being unity) will be $\frac{M-F}{M+W}$; hence, by the laws of uniformly accelerated motions, $\frac{M-F}{M+W} \times rt^2 = S$, confequently $F = M - \frac{\overline{M+W} \times S}{rt^2}$. To exemplify this, let us take the cafe of the laft experiment, where $M = 7$, $W = 25\frac{3}{4}$, $S = 4\frac{7}{12}$ feet, $t = 4''$; hence $F = 7 - \frac{32\frac{3}{4} \times 4\frac{7}{12}}{16\frac{1}{12} \times 16} = 6.417$; confequently the friction was to the weight of the rubbing body as 6.4167 to 25.75. And the great accuracy of determining the friction by this method is manifeft from hence, that if an error of 1 inch had been made in the defcent (and experiments carefully made may always determine the fpace to a much greater exactnefs) it would not have affected the conclufion $\frac{1}{600}$dth part of the whole.

5. We come in the next place to determine, whether friction, *cæteris paribus*, varies in proportion to the weight or preffure. Now if the whole quantity of the friction of a body, meafured by a weight without inertia equivalent to the friction drawing the body backwards, increafes in proportion to its weight, it is manifeft, that the retardation of the velocity of the body arifing from the friction will not be altered; for the

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retardation varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$; hence, if a body be put in motion upon the horizontal plane by any moving force, if both the weight of the body and the moving force be increaſed in the ſame ratio, the acceleration ariſing from that moving force will remain the ſame, becauſe the accelerative force varies as the moving force divided by the whole quantity of matter, and both are increaſed in the ſame ratio; and if the quantity of friction increaſes alſo as the weight, then the retardation ariſing from the friction will, from what has been ſaid, remain the ſame, and therefore the whole acceleration of the body will not be altered; conſequently the body ought, upon this ſuppoſition, ſtill to deſcribe the ſame ſpace in the ſame time. Hence, by obſerving the ſpaces deſcribed in the ſame time, when both the body and the moving force are increaſed in the ſame ratio, we may determine whether the friction increaſes in proportion to the weight. The following experiments were therefore made in order to aſcertain this matter.

Exp. 1. A body weighing 10 oz. by a moving force of 4 oz. deſcribed in 2″ a ſpace of 51 inches; by loading the body with 10 oz. and the moving force with 4 oz. it deſcribed 56 inches in 2″; and by loading the body again with 10 oz. and the moving force with 4 oz. it deſcribed 63 inches in 2″.

Exp. 2. A body, whoſe weight was 16 oz. by a moving force of 5 oz. deſcribed a ſpace of 49 inches in 3″; and by loading the body with 64 oz. and the moving force with 20 oz. the ſpace deſcribed in the ſame time was 64 inches.

Exp. 3. A body weighing 6 oz. by a moving force of 2½ oz. deſcribed 28 inches in 2″; and by loading the body with 24 oz. and the moving force with 10 oz. the ſpace deſcribed in the ſame time was 54 inches.

EXP. 4. A body weighing 8 oz. by a moving force of 4 oz. described $33\frac{1}{2}$ inches in 2″; and by loading the body with 8 oz. and the moving force with 4 oz. the space described in the same time was 47 inches.

EXP. 5. A body whose weight was 9 oz. by a moving force of $4\frac{1}{2}$ oz. described 48 inches in 2″; and by loading the body with 9 oz. and the moving force with $4\frac{1}{2}$ oz. the space described in the same time was 60 inches.

EXP. 6. A body weighing 10 oz. by a moving force of 3 oz. described 20 inches in 2″; by loading the body with 10 oz. and the moving force with 3 oz. the space described in the same time was 31 inches; and by loading the body again with 30 oz. and the moving force with 9 oz. the space described was 34 inches in 2″.

From these experiments, and many others which it is not necessary here to relate, it appears, that the space described is always increased by increasing the weight of the body and the accelerative force in the same ratio; and as the acceleration arising from the moving force continued the same, it is manifest, that the retardation arising from the friction must have been diminished, for the whole accelerative force must have been increased on account of the increase of the space described in the same time; and hence (as the retardation from friction varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$) *the quantity of friction increases in a less ratio than the quantity of matter or weight of the body.*

6. We come now to the last thing which it was proposed to determine, that is, whether the friction varies by varying the surface on which the body moves. Let us call two of the surfaces A and *a*, the former being the greater, and the latter the less. Now the weight on every given part of *a* is as much greater than

than the weight on an equal part of A, as A is greater than *a*; if therefore the friction was in proportion to the weight, *cæteris paribus*, it is manifest, that the friction on *a* would be equal to the friction on A, the whole friction being, upon such a supposition, as the weight on any given part of each surface multiplied into the number of such parts or into the whole area, which products, from the proportion above, are equal. But from the last experiments it has been proved, that the friction on any given surface increases in a less ratio than the weight; consequently the friction on any given part of *a* has a less ratio to the friction on an equal part of A than A has to *a*, and hence the friction on *a* is less than the friction on A, that is, the smallest surface has always the least friction. But as this conclusion is contrary to the generally received opinion, I have thought it proper to confirm the same by a set of experiments. But before I proceed to relate them, I will beg leave to recommend to those, who may afterwards be induced to repeat them, the following cautions, which are extremely necessary to be attended to. Great care must be taken that the two surfaces have exactly the same degree of roughness; in order to be certain of which, such bodies must be chosen as have no knots in them, and whose grain is so very regular that when the two surfaces are planed with a fine rough plane, their roughness may be the same, which will not be the case if the body be knotty, or the grain irregular, or if it happens not to run in the same direction on both surfaces. When you cannot depend on the surfaces having the same degree of roughness, the best way will be to paste some fine rough paper on each surface, which perhaps will give a more equal degree of roughness than can be obtained by any other method. Now as the proof which I have already given depends only on the motion

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of the body upon the *same* surface, it is not liable to any inaccuracy of the kind which the preceding cautions have been given to avoid, nor indeed to any other, and therefore it must be perfectly conclusive. In the following experiments the cautions mentioned above were carefully attended to.

EXP. 1. A body was taken whose flat surface was to its edge as 22 : 9, and with the same moving force the body described on its flat side $33\frac{1}{2}$ inches in $2''$, and on its edge 47 inches in the same time.

EXP. 2. A second body was taken whose flat surface was to its edge as 32 : 3, and with the same moving force it described on its flat side 32 inches in $2''$, and on its edge it described $37\frac{1}{2}$ inches in the same time.

EXP. 3. I took another body and covered one of its surfaces, whose length was 9 inches, with a fine rough paper, and by applying a moving force, it described 25 inches in $2''$; I then took off some paper from the middle, leaving only $\frac{3}{8}$ of an inch at the two ends, and with the same moving force it described 40 inches in the same time.

EXP. 4. Another body was taken which had one of its surfaces, whose length was 9 inches, covered with a fine rough paper, and by applying a moving force it described 42 inches in $2''$; some of the paper was then taken off from the middle, leaving only $1\frac{3}{8}$ inches at the two ends, and with the same moving force it described 54 inches in $2''$; I then took off more paper, leaving only $\frac{1}{4}$ of an inch at the two ends, and the body then described, by the same moving force, 60 inches in the same time.

In the two last experiments the paper which was taken off the surface was laid on the body, that its weight might not be altered.

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Exp. 5. A body was taken whoſe flat ſurface was to its edge as 30 : 17; the *flat* ſide was laid upon the horizontal plane, a moving force was applied, and the ſtage was fixed in order to ſtop the moving force, in conſequence of which the body would then go on with the velocity acquired until the friction had deſtroyed all its motion; when it appeared from a mean of 12 trials that the body moved, after its acceleration ceaſed, $5\frac{2}{5}$ inches before it ſtopped. The *edge* was then applied, and the moving force deſcended through the ſame ſpace, and it was found, from a mean of the ſame number of trials, that the ſpace deſcribed was $7\frac{1}{3}$ inches before the body loſt all its motion, after it ceaſed to be accelerated.

Exp. 6. Another body was then taken whoſe flat ſurface was to its edge as 60 : 19, and, by proceeding as before, on the flat ſurface it deſcribed, at a mean of 12 trials, $5\frac{1}{8}$ inches, and on the edge $6\frac{1}{2}\frac{1}{4}$ inches, before it ſtopped, after the acceleration ceaſed.

Exp. 7. Another body was taken whoſe flat ſurface was to its edge as 26 : 3, and the ſpaces deſcribed on theſe two ſurfaces, after the acceleration ended, were, at a mean of 10 trials, $4\frac{3}{7}$ and $7\frac{7}{10}$ inches reſpectively.

From all theſe different experiments it appears, that the ſmalleſt ſurface had always the leaſt friction, which agrees with the conſequence deduced from the conſideration that the friction does not increaſe in ſo great a ratio as the weight; we may therefore conclude, that *the friction of a body does not continue the ſame when it has different ſurfaces applied to the plane on which it moves, but that the ſmalleſt ſurface will have the leaſt friction.*

7. Having thus eſtabliſhed, from the moſt deciſive experiments, all that I propoſed relative to friction, I think it proper, before

before I conclude, to give the reſult of my examination into the nature of the experiments which have been made by others; which were repeated, in order to ſee how far they were concluſive in reſpect to the principles which have been deduced from them. The experiments which have been made by all the authors that I have ſeen, have been thus inſtituted. To find what moving force would *juſt* put a body at reſt in motion: and they concluded from thence, that the accelerative force was then equal to the friction; but it is manifeſt, that any force which will put a body in motion muſt be *greater* than the force which oppoſes its motion, otherwiſe it could not overcome it; and hence, if there were no other objection than this, it is evident, that the friction could not be very accurately obtained; but there is another objection which totally deſtroys the experiment ſo far as it tends to ſhow the quantity of friction, which is the ſtrong coheſion of the body to the plane when it lies at reſt; and this is confirmed by the following experiments. 1ſt, A body of 12¾ oz. was laid upon an horizontal plane, and then loaded with a weight of 8 lb. and ſuch a moving force was applied as would, when the body was juſt *put* in motion, continue that motion without any acceleration, in which caſe the friction muſt be juſt equal to the accelerative force. The body was then ſtopped, when it appeared, that the ſame moving force which had *kept* the body in motion before, would not *put* it in motion, and it was found neceſſary to take off 4½ oz. from the body before the ſame moving force *would* put it in motion; it appears, therefore, that this body, when laid upon the plane at reſt, acquired a very ſtrong coheſion to it. 2dly, A body whoſe weight was 16 oz. was laid at reſt upon the horizontal plane, and it was found that a moving force of 6 oz. would juſt *put* it in motion; but that a moving force of 4 oz.

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would, when it was juſt put in motion, *continue* that motion without any acceleration, and therefore the accelerative force muſt *then* have been equal to the friction, and not when the moving force of 6 oz. was applied.

From theſe experiments therefore it appears, how very conſiderable the coheſion was in proportion to the friction when the body was in motion; it being, in the latter caſe, almoſt $\frac{1}{3}$, and in the former it was found to be very nearly equal to the whole friction. All the concluſions therefore deduced from the experiments, which have been inſtituted to determine the friction from the force neceſſary to *put* a body in motion (and I have never ſeen any deſcribed but upon ſuch a principle) have manifeſtly been totally falſe; as ſuch experiments only ſhew the reſiſtance which ariſes from the coheſion and friction conjointly.

8. I ſhall conclude this part of the ſubject with a remark upon Art. 5. It appears from all the experiments which I have made, that the proportion of the increaſe of the friction to the increaſe of the weight was different in all the different bodies which were made uſe of; no general rule therefore can be eſtabliſhed to determine this for *all* bodies, and the experiments which I have hitherto made have not been ſufficient to determine it for the *ſame* body. At ſome future opportunity, when I have more leiſure, I intend to repeat the experiments in order to eſtabliſh, in ſome particular caſes, the law by which the quantity of friction increaſes by increaſing the weight. Leaving this ſubject therefore for the preſent, I ſhall proceed to eſtabliſh a theory upon the principles which we have already deduced from our experiments.

PROPOSITION I.

Let e, f, g, (fig. 1.) *represent either a cylinder, or that circular section of a body on which it rolls down the inclined plane* CA *in consequence of its friction, to find the time of descent and the number of revolutions.*

As it has been proved in Art. 5. that the friction of a body does not increase in proportion to its weight or pressure, we cannot therefore, by knowing the friction on any other plane, determine the friction on CA; the friction therefore on CA can only be determined by experiments made upon *that* plane, that is, by letting the body descend from rest, and observing the space described in the first second of time; call that space a, and then, as by Art. 3. friction is a uniformly retarding force, the body must be uniformly accelerated, and consequently the whole time of descent in seconds will be $= \sqrt{\frac{AC}{a}}$. Now to determine the number of revolutions, let s be the center of oscillation to the point of suspension a *; then, because no force acting at a can affect the motion of the point s, that point, notwithstanding the action of the friction at a, will always have a motion parallel to CA uniformly accelerated by a force equal to that with which the body would be accelerated if it had no friction; hence, if $2m = 32\frac{1}{6}$ feet, the velocity acquired by the point s in the first second will be $= \frac{2m \times CB}{CA}$; now the excess of the velocity

* a and s are not fixed points in the body, but the former always represents that point of the body in contact with the plane, and the latter the corresponding center of oscillation.

locity of the point s above that of r (r being the center) is manifestly the velocity with which s is carried about r; hence the velocity of s about the center $=\frac{2m\times CB}{CA}-2a=\frac{2m\times CB-2a\times CA}{CA}$, consequently $rs : ra :: \frac{2m\times CB-2a\times CA}{CA} : \frac{2m\times ra\times CB-2a\times ra\times CA}{rs\times CA}$ = the velocity with which a point of the circumference is carried about the center, and which therefore expresses the force which accelerates the rotation; now as $2a$ expresses the accelerative force of the body down the plane, and the spaces described in the same time are in proportion to those forces, we have $2a : CA :: \frac{2m\times ra\times CB-2a\times ra\times CA}{rs\times CA} : \frac{m\times ra\times CB-a\times ra\times CA}{a\times rs}$ the space which any point of the circumference describes about the center in the whole time of the body's descent down CA; which being divided by the circumference $p\times ra$ (where $p=6.283$ &c.) will give $\frac{m\times BC-a\times AC}{p\times a\times rs}$ for the whole number of revolutions required.

Cor. 1. If $a\times CA=m\times BC$, the number of revolutions $=0$, and therefore the body will then only slide; consequently the friction vanishes.

Cor. 2. Let $a'r's'$ (fig. 2.) be the next position of ars, and draw $tr'b$ parallel to sa, then will $s't$ represent the retardation of the center r arising from friction, and $a'b$ will represent the acceleration of a point of the circumference about its center; hence the retardation of the center : acceleration of the circumference about the center :: $s't : a'b$ :: (by sim. Δ's) $tr' : br' :: rs : ra$.

Cor. 3. If a' coincides with a, the body does not *slide* but only *roll*; now in this case $ss' : rr' :: as : ar$; but as ss' and rr' represent the ratio of the velocities of the points s and r,

they will be to each other as $\frac{2m \times BC}{CA} : 2a$ or as $m \times CB : a \times CA$; hence, when the body *rolls* without *ſliding*, $as : ar :: m \times CB : a \times CA$.

Cor. 4. The time of deſcent down CA is $= \sqrt{\frac{AC}{a}}$; but by the laſt Cor. when the body *rolls* without *ſliding*, $a = \frac{m \times ra \times BC}{sa \times AC}$, hence the time of deſcent in that caſe $= AC\sqrt{\frac{sa}{m \times ra \times BC}}$; now the time of deſcent, if there were no friction, would be $= \frac{AC}{\sqrt{m \times BC}}$, hence the time of deſcent, when the body *rolls* without *ſliding* : time of free deſcent :: $\sqrt{sa} : \sqrt{ra}$.

Cor. 5. By the laſt Cor. it appears, that when the body juſt *rolls* without *ſliding*, or when the friction is juſt equal to the accelerative force, the time of deſcent $= AC\sqrt{\frac{sa}{m \times ra \times BC}}$; now it is manifeſt, that the time of deſcent will continue the ſame, if the friction be increaſed, for the body will ſtill freely roll, as no increaſe of the friction acting at a can affect the motion of the point s.

If the body be projected from C with a velocity, and at the ſame time have a rotatory motion, the time of deſcent and the number of revolutions may be determined from the common principles of uniformly accelerated motions, as we have already inveſtigated the accelerative force of the body down the plane and of its rotation about its axis; it ſeems therefore unneceſſary to lengthen out this paper with the inveſtigations.

PROPOSITION II.

Let the body be projected on an horizontal plane LM (fig. 3.) *with a given velocity, to determine the ſpace through which the body will move before it ſtops, or before its motion becomes uniform.*

CASE I. 1. Suppoſe the body to have no rotatory motion when it begins to move; and let $a=$ the velocity of projection *per* ſecond meaſured in feet, and let the retarding force of the friction of the body, meaſured by the velocity of the body which it can deſtroy in one ſecond of time, be determined by experiment and called F, and let x be the ſpace through which the body would move by the time its motion was all deſtroyed when projected with the velocity a, and retarded by a force F; then, from the principles of uniformly retarded motion, $x=\frac{a^2}{2F}$, and if $t=$ time of deſcribing that ſpace, we have $t=\frac{a}{F}$, and hence the ſpace deſcribed in the firſt ſecond of time $=\frac{2a-F}{2}$. Now it is manifeſt, that when the rotatory motion of the body about its axis is equal to its progreſſive motion, the point a will be carried backwards by the *former* motion as much as it is carried forwards by the *latter*; conſequently the point of contact of the body with the plane will then have no motion in the direction of the plane, and hence the friction will at that inſtant ceaſe, and the body will continue to *roll* on uniformly without *ſliding* with the velocity which it has at that point. Put therefore $z=$ the ſpace deſcribed from the commencement of the motion till it becomes uniform, then the body being uniformly retarded, the ſpaces from the end of

the motion vary as the squares of the velocities, hence $\frac{a^2}{2F} : a^2 (:: 1 : 2F) :: \frac{a^2}{2F} - z : a^2 - 2Fz =$ square of the progressive velocity when the motion becomes uniform; therefore the velocity destroyed by friction $= a - \sqrt{a^2 - 2Fz}$; hence, as the velocity generated or destroyed in the same time is in proportion to the force, we have by Cor. 2. Prop. 1. $rs : ra :: a - \sqrt{a^2 - 2Fz} : \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}}$ the velocity of the circumference efg generated about the center, consequently $\sqrt{a^2 - 2Fz} = \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}}$, and hence $z = \frac{\overline{rs^2 + 2rs \times ra} \times a^2}{as^2 \times 2F}$ the space which the body describes before the motion becomes uniform.

2. If we substitute this value of z into the expression for the velocity, we shall have $a \times \frac{ra}{rs}$ for the velocity of the body when its motion becomes uniform; hence therefore it appears, that the velocity of the body, when the friction ceases, will be the same whatever be the quantity of the friction. If the body be the circumference of a circle, it will always lose half the velocity before its motion becomes uniform.

CASE II. 1. Let the body, besides having a progressive velocity in the direction LM (fig. 3.) have also a rotatory motion about its center in the direction gfe, and let v represent the initial velocity of any point of the circumference about the center, and suppose it first to be less than a; then friction being a uniformly retarding force, no alteration of the velocity of the point of contact of the body upon the plane can affect the quantity of friction; hence the progressive velocity of the body will be the same as before, and consequently the rotatory velocity

city generated by friction will alſo be the ſame, to which if we add the velocity about the center at the beginning of the motion, we ſhall have the whole rotatory motion; hence therefore, $v + \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}} = \sqrt{a^2 - 2Fz}$, conſequently $z = \frac{a^2 \times as^2 - \overline{v \times rs + a \times ra}^2}{2F \times as^2}$ the ſpace deſcribed before the motion becomes uniform.

2. If this value of z be ſubſtituted into the expreſſion for the velocity, we ſhall have $\frac{v \times rs + a \times ra}{as}$ for the velocity when the friction ceaſes.

3. If $v = a$, then $z = o$, and hence the body will continue to move uniformly with the firſt velocity.

4. If v be greater than a, then the rotatory motion of the point a on the plane being greater than its progreſſive motion and in a contrary direction, the abſolute motion of the point a upon the plane will be in the direction ML, and conſequently friction will now act in the direction LM in which the body moves, and therefore will accelerate the *progreſſive* and retard the *rotatory* motion; hence it appears, that *the progreſſive motion of a body may be* ACCELERATED *by friction.* Now to determine the ſpace deſcribed before the motion becomes uniform, we may obſerve, that as the progreſſive motion of the body is now accelerated, the velocity after it has deſcribed any ſpace z will be $= \sqrt{a^2 + 2Fz}$, hence the velocity acquired $= \sqrt{a^2 + 2Fz} - a$, and conſequently the rotatory velocity deſtroyed $\frac{ra}{rs} \times \overline{\sqrt{a^2 + 2Fz} - a}$, hence $v - \frac{ra}{rs} \times \overline{\sqrt{a^2 + 2Fz} - a} = \sqrt{a^2 + 2Fz}$, therefore $z = \frac{\overline{rs \times v + ra \times a}^2 - a^2 \times as^2}{2F \times as^2}$ the ſpace required.

5. If

5. If $a=o$, or the body be placed upon the plane without any progreſſive velocity, then $z=\frac{rs^2 \times v^2}{2F \times as^2}$.

CASE III. 1. Let the given rotatory motion be in the direction gef; then as the friction muſt in this caſe always act in the direction ML, it muſt continually tend to deſtroy both the progreſſive and rotatory motion. Now as the velocity deſtroyed in the ſame time is in proportion to the retarding force, and the force which retards the *rotatory* is to the force which retards the *progreſſive* velocity by Cor. 2. Prop. 1. as $ra : rs$, therefore if v be to a as ra is to rs, then the retarding forces being in proportion to the velocities, both motions will be deſtroyed together, and conſequently the body, after deſcribing a certain ſpace, will reſt; which ſpace, being that deſcribed by the body uniformly retarded by the force F, will, from what was proved in Caſe I. be equal to $\frac{a^2}{2F}$.

2. If v bears a greater proportion to a than ra does to rs, it is manifeſt, that the rotatory motion will not be all deſtroyed when the progreſſive is; conſequently the body, after it has deſcribed the ſpace $\frac{a^2}{2F}$, will return back in the direction ML; for the progreſſive motion being then deſtroyed, and the rotatory motion ſtill continuing in the direction gef, will cauſe the body to return with an accelerative velocity until the friction ceaſes by the body's beginning to roll, after which it will move on uniformly. Now to determine the ſpace deſcribed before this happens, we have $rs : ra :: a : \frac{ra \times a}{rs}$ the rotatory velocity deſtroyed when the progreſſive is all loſt; hence $v - \frac{ra \times a}{rs} = \frac{v \times rs - a \times ra}{rs} =$ the rotatory velocity at that time, which

being

being fubftituted for v in the laft article of Cafe II. gives $\frac{\overline{v+rs-a\times ra}^2}{2F\times as^2}$ for the fpace defcribed before the motion becomes uniform.

3. If v has a lefs proportion to a than ra has to rs, it is manifeft, that the *rotatory* motion will be deftroyed before the *progreffive*; in which cafe a rotatory motion will be generated in a contrary direction until the two motions become equal, when the friction will inftantly ceafe, and the body will then move on uniformly. Now $ra : rs :: v : \frac{v\times rs}{ra}$ the progreffive velocity deftroyed when the rotatory velocity ceafes, hence $a-\frac{v\times rs}{ra}=\frac{a\times ra-v\times rs}{ra}=$ progreffive velocity when it begins its rotatory motion in a contrary direction; fubftitute therefore this quantity for a in the expreffion for z in Cafe I. and we have $\frac{\overline{rs^2+2rs\times ra}\times\overline{a\times ra-v\times rs}^2}{as^2\times ar^2\times 2F}$ for the fpace defcribed after the rotatory motion ceafes before the motion of the body becomes uniform. Now to determine the fpace defcribed before the rotatory motion was all deftroyed, we have (as the fpace from the end of a uniformly retarded motion varies as the fquare of the velocity) $a^2 : \frac{a^2}{2F} :: \frac{\overline{a\times ra-v\times rs}^2}{ra^2} : \frac{\overline{a\times ra-v\times rs}^2}{2F\times ra^2}$ the fpace that could have been defcribed from the time that the rotatory velocity was deftroyed, until the progreffive motion would have been deftroyed had the fricton continued to act; hence $\frac{a^2}{2F}-\frac{\overline{a\times ra-v\times rs}^2}{2F\times ra^2}=\frac{2av\times ra\times rs-v^2\times rs^2}{2F\times ra^2}=$ the fpace defcribed when the rotatory motion was all deftroyed, hence

$$\frac{\overline{rs^2+2rs\times ra}\times\overline{a\times ra-v\times rs}^2}{as^2\times ar^2\times 2F}+\frac{2av\times ra\times rs-v^2\times rs^2}{2F\times ra^2}=$$ whole fpace defcribed by the body before its motion becomes uniform.

DEFINITION.

The CENTER *of* FRICTION *is that point in the base of a body on which it revolves, into which if the whole surface of the base, and the mass of the body were collected, and made to revolve about the center of the base of the given body, the angular velocity destroyed by its friction would be equal to the angular velocity destroyed in the given body by its friction in the same time.*

PROPOSITION III.

To find the center of friction.

Let FGH (fig. 4.) be the base of a body revolving about its center C, and suppose about a, b, c, &c. to be indefinitely small parts of the base, and let A, B, C, &c. be the corresponding parts of the solid, or the prismatic parts having a, b, c, &c. for their bases; and P the center of friction. Now it is manifest, that the decrement of the angular velocity must vary as the whole diminution of the momentum of rotation caused by the friction *directly*, and as the whole momentum of rotation or effect of the inertia of all the particles of the solid *inversely*; the *former* being employed in diminishing the angular velocity, and the *latter* in opposing that diminution by the endeavour of the particles to persevere in their motion. Hence, if the effect of the friction varies as the effect of the inertia, the decrements of the angular velocity in a given time will be equal. Now as the quantity of friction (as has been proved from experiments) does not depend on the velocity, the effect of the friction of the elementary parts of the base a, b, c, &c.

 will

will be as $a \times aC$, $b \times bC$, $c \times cC$, &c. alſo the effect of the inertia of the correſponding parts of the body will be as $A \times aC^2$, $B \times bC^2$, $C \times cC^2$, &c. Now when the whole ſurface of the baſe and maſs of the body are concentrated in P, the effect of the friction will be as $\overline{a+b+c+\&c.} \times CP$, and of the inertia as $\overline{A+B+C+\&c.} \times CP^2$; conſequently $a \times aC + b \times bC + c \times cC + \&c. : \overline{a+b+c+\&c.} \times CP :: A \times aC^2 + B \times bC^2 + C \times cC^2 + \&c. : \overline{A+B+C+\&c.} \times CP^2$; and hence

$$CP = \frac{\overline{A \times aC^2 + B \times bC^2 + C \times cC^2 + \&c.} \times \overline{a+b+c+\&c.}}{\overline{a \times aC + b \times bC + c \times cC + \&c.} \times \overline{A+B+C+\&c.}} =$$ (if $S =$ the ſum of the products of each particle into the ſquare of its diſtance from the axis of motion, $T =$ the ſum of the products of each part of the baſe into its diſtance from the center, $s =$ the area of the baſe, $t =$ the ſolid content of the body) $\frac{S \times s}{T \times t}$.

PROPOSITION IV.

Given the velocity with which a body begins to revolve about the center of its baſe, to determine the number of revolutions which the body will make before all its motion be deſtroyed.

Let the friction, expreſſed by the velocity which it is able to deſtroy in the body if it were projected in a right line horizontally in one ſecond, be determined by experiment, and called F; and ſuppoſe the initial velocity of the center of friction P about C to be a. Then conceiving the whole ſurface of the baſe and maſs of the body to be collected into the point P, and (as has been proved in Prop. II.) $\frac{a^2}{2F}$ will be the ſpace which the body ſo concentrated will deſcribe before all its motion be deſtroyed;

hence if we put $z=PC$, $p=$ the circumference of a circle whoſe radius is unity, then will $pz=$ circumference deſcribed by the point P; conſequently $\frac{a^2}{2pzF}=$ the number of revolutions required.

Cor. If the ſolid be a cylinder and r be the radius of its baſe, then $z=\frac{3r}{4}$, and therefore the number of revolutions $=\frac{2a^2}{3prF}$.

PROPOSITION V.

To find the nature of the curve deſcribed by any point of a body affected by friction, when it deſcends down any inclined plane.

Let efg (fig. 5.) be the body, the points a, r, s, as in Prop. I. and conceive st, rn, to be two indefinitely ſmall ſpaces deſcribed by the points s and r in the ſame time, and which therefore will repreſent the velocities of thoſe points; but from Prop. I. the ratio of theſe velocities is expreſſed by $m\times CB : a\times CA$, hence $st : rn :: m\times CB : a\times CA$. With the center r let a circle vw be deſcribed touching the plane LM which is parallel to AC at the point b, and let the radius of this circle be ſuch that, conceiving it to deſcend upon the plane LM along with the body deſcending on CA, the point b may be at reſt, or the circle may roll without ſliding. To determine which radius, produce rs to x, parallel to which draw ndy, and produce nt to z; now it is manifeſt, that in order to anſwer the conditions above-mentioned, the velocity of the point x muſt be to the velocity of the point r as $2:1$, that is, $zx : yx :: 2:1$, hence $zy=yx=nr$. Now $zy : dt$ ($:: ny : nd$) $:: rx : rs$; therefore $dt=\frac{rs}{rx}\times zy=\frac{rs}{rx}\times nr$, hence ts ($=td+ds=td+nr=$

Fig. 1.

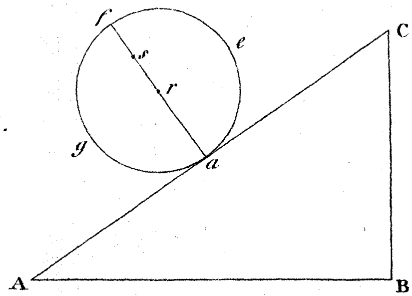


Fig. 2.

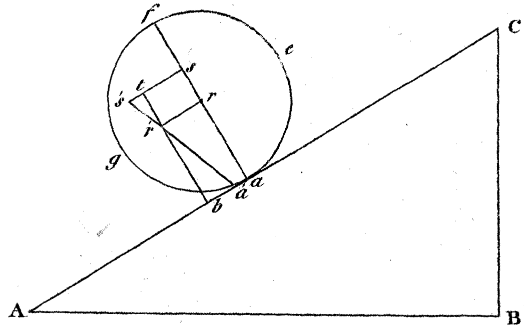


Fig. 3.

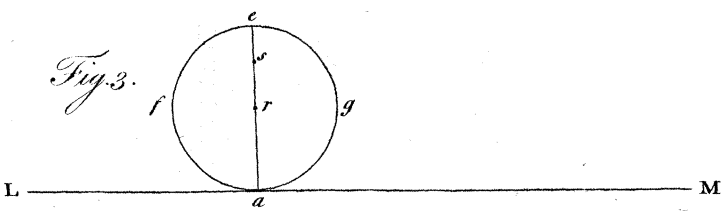


Fig. 4.

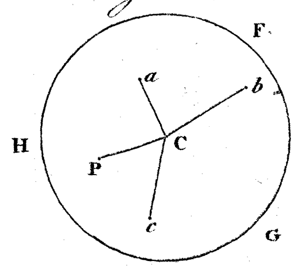
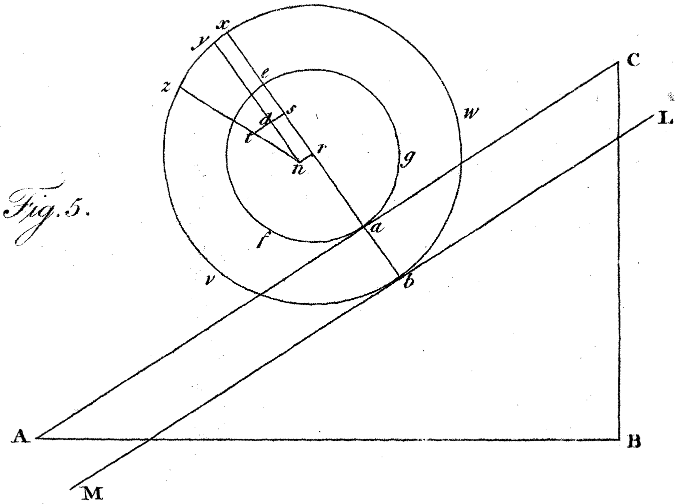


Fig. 5.



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$\frac{rs}{rx} \times nr + nr) = \frac{rs+rx}{rx} \times nr$, consequently $\frac{rs+rx}{rx} : 1 :: ts : nr ::$ (from what is proved above) $m \times CB : a \times CA$; therefore $a \times CA \times rs + a \times CA \times rx = m \times CB \times rx$, hence $rx = \frac{a \times CA \times rs}{m \times CB - a \times CA}$ the radius of the circle which rolling down the inclined plane LM, and carrying the body with it, will give the true ratio of its progressive to its rotatory motion, and consequently that point of the circle which coincides with any given point of the body will, as the circle revolves upon the line LM, describe the same curve as the corresponding point of the body; but as the nature of the curve described by any point of a circle revolving upon a straight line is already very well known, it seems unnecessary to give the investigation.

By a method of reasoning, not very different, may the nature of the curve, which is described by any point of a body moving upon an horizontal plane, and affected by friction, be determined.